

ON CENTRAL AUTOMORPHISMS OF GROUPS AND NILPOTENT RINGS

YASSINE GUERBOUSSA AND BOUNABI DAOUD

ABSTRACT. Let G be a group. The central automorphism group $Aut_c(G)$ of G is the centralizer of $Inn(G)$ the subgroup of $Aut(G)$ of inner automorphisms. There is a one to one map $\sigma \mapsto h_\sigma$ from the set $Aut_c(G)$ onto the set $Hom(G, Z(G))$ of homomorphisms from G onto its center, with $h_\sigma(x) = x^{-1}\sigma(x)$. This map can be used to obtain informations about the size of $Aut_c(G)$, and also about its structure in some special cases. In this paper we see how to use it to obtain informations about the structure of $Aut_c(G)$ in the general case. The notion of the adjoint group of a ring is the main tool in our approach.

1. Introduction

It is very difficult to prove general theorems about the automorphisms of finite p-groups, and very little is known about them. An automorphism of a group G is termed central if it commutes with every inner automorphism, clearly the central automorphisms of G form a normal subgroup $Aut_c(G)$ of $Aut(G)$. If G is a finite p -group, then $Aut_c(G)$ has a great importance in investigating $Aut(G)$, and it has been studied by several authors, see for instance ([2]-[5], and also [9], [10]).

It is easy to see that the map, or the Adney-Yen map for convenience, $\sigma \mapsto h_\sigma$ determines a one to one map from the set $Aut_c(G)$ onto the set $Hom(G, Z(G))$, where $h_\sigma(x) = x^{-1}\sigma(x)$. What are the informations that can be deduced about $Aut_c(G)$ from this relation? this is the main task of this paper.

Let R be a (associative) ring. Under the circle composition $x \circ y = x + y + xy$, the set of all elements of R forms a monoid with identity element $0 \in R$, this monoid is called the adjoint monoid or semigroup of the ring R . The adjoint group R° of R is the group of invertible elements in this monoid.

Let consider the set $Hom(G, Z(G))$ as a ring, the addition is defined in the usual way and we take the composition of maps as a multiplication. Our main observation is that the Adney-Yen map defines an isomorphism between $Aut_c(G)$ and the adjoint group of the ring $Hom(G, Z(G))$.

When the ring R has an identity 1, the mapping $x \mapsto 1 + x$ determines a group isomorphism from R° to the multiplicative group of the ring R . This agrees with the usual case when G is abelian : the central automorphism group coincides with $Aut(G)$ which is the multiplicative group of the ring $End(G)$.

Assume that G is finite. It was proved in [2] that the Adney-Yen map is a bijection if G does not have a non-trivial abelian direct factor. In the light of our observation, this is equivalent to saying that $Hom(G, Z(G))$ is a radical ring. Following Jacobson, a ring R is termed radical if its adjoint semigroup is a group, or equivalently $R^\circ = R$. Adjoint groups of radical rings are interesting objects

to study and we may find a considerable number of papers in the subject (see [6] for some references). The above results and some of its consequences are discussed in Section 2 in a more general context. And since we are mainly interested to finite p -groups, the remaining sections are devoted to their central automorphisms, in Section 3 we introduce the notion of a p -nil ring in order to studying the structure of $Aut_c(G)$ when G is a finite p -group with $Z(G) \leq \Phi(G)$. The results of this section are applied in Section 4 to the longstanding problem of whether every non-abelian finite p -group has a non-inner automorphism of order p (see [1]), we give a necessary and a sufficient condition for a finite p -groups to have a non-inner central automorphism of order $p > 2$.

Throughout, the unexplained notation is standard in the literature. We denote by $Hom(G, N)$ the group of homomorphisms from G to an abelian group N . We denote by $d(G)$ the minimal number of generators of G , and the rank $r(G)$ of G is defined to be $\sup\{d(H), H \leq G\}$. The exponent of G is denoted by $exp(G)$ and \mathbb{Z}_n denotes the ring of integers modulo n .

Lemma 1.1. *If M and N are finite abelian p -groups, then the rank and the exponent of the abelian group $Hom(M, N)$ are equal respectively to $r(M).r(N)$ and $\min\{exp(M), exp(N)\}$.*

Proof. This follows immediately from the properties

$$Hom(\prod_i M_i, \prod_j N_j) \cong \prod_{i,j} Hom(M_i, N_j)$$

where M_i and N_j are abelian groups, and

$$Hom(\mathbb{Z}_{p^n}, \mathbb{Z}_{p^m}) \cong \mathbb{Z}_{p^{\min\{n, m\}}}.$$

□

Given an associative ring R , we denote by R^+ the additive group of R . The n th power R^n of R is the additive group generated by all the products of n elements of R . We say that R is nilpotent if $R^{n+1} = 0$ for some non-negative integer n , the least integer n satisfying $R^{n+1} = 0$ is called the class of nilpotency of the ring R . Note that every nilpotent ring R is radical since for every $x \in R$ we have

$$x \circ \sum_i (-1)^i x^i = (\sum_i (-1)^i x^i) \circ x = 0.$$

The Jacobson radical of the ring R is the largest ideal of R contained in the adjoint group R° . This implies that R is radical if and only if it coincides with its Jacobson radical. By a classical result the Jacobson radical of an artinian ring is nilpotent, so every artinian (in particular finite) radical ring is nilpotent.

The following lemma is standard in the literature (see [8], Section I.6).

Lemma 1.2. *The adjoint group of a nilpotent ring R is nilpotent of class at most equals to the nilpotency class of R .*

Proof. The series of ideals

$$R \supset R^2 \supset \dots \supset R^{n+1} = 0$$

induces a central series in the adjoint group of the ring R .

□

The following lemma is a variant of theorem B in [6], it gives a bound for the rank of the adjoint group of a finite (periodic in general) radical ring R in term of the rank of its additive group.

Lemma 1.3. *Let R be a finite radical ring. Then $r(R^\circ) \leq 3r(R^+)$, and if the order of R is odd then $r(R^\circ) \leq 2r(R^+)$.*

2. Central automorphisms and radical rings

We begin with the following general remark. Every abelian normal subgroup A of a group G can be viewed as a G -module via conjugation $a^x = x^{-1}ax$, with $x \in G$ and $a \in A$. A derivation of G into A is a mapping $\delta : G \rightarrow A$ such that $\delta(xy) = \delta(x)^x \delta(y)$. The set $Der(G, A)$ of these derivations is a ring under the addition $\delta_1 + \delta_2(x) = \delta_1(x) + \delta_2(x)$ and the multiplication $\delta_1 \delta_2(x) = \delta_2(\delta_1(x))$, with $\delta_1, \delta_2 \in Der(G, A)$ and $x \in G$. Let denote by $End_A(G)$ the set of endomorphisms u of G having the property $x^{-1}u(x) \in A$, for all $x \in G$. We check easily that $End_A(G)$ is a submonoid of $End(G)$ and every endomorphism $u \in End_A(G)$ defines a derivation $\delta_u(x) = x^{-1}u(x)$ of G into A . Note also that to each derivation $\delta \in Der(G, A)$ we can associate an endomorphism $u \in End_A(G)$ with $u(x) = x\delta(x)$.

Lemma 2.1. *Under the above notation, the mapping $u \mapsto \delta_u$ is an isomorphism between the monoid $End_A(G)$ and the adjoint monoid of the ring $Der(G, A)$. In particular it induces an isomorphism between the corresponding groups of invertible elements.*

Proof. Straightforward verification. □

Since the center $Z(G)$ is a trivial G -module, we have $Der(G, Z(G)) = Hom(G, Z(G))$. So for $A = Z(G)$ the mapping defined above reduces to the Adney-Yen map. It follows that

Proposition 2.2. *The Adney-Yen map determines an isomorphism between the central automorphism group $Aut_c(G)$ and the adjoint group of the ring $Hom(G, Z(G))$.*

Assume that G is finite. In [2] Adney and Yen have proved that every endomorphism in $End_{Z(G)}(G)$ is an automorphism if and only if G is purely non-abelian, that is G does not have a non-trivial abelian direct factor. The above observation allows us to set this result under the form

Theorem 2.3. (Adney-Yen) *Let G be a finite group. Then the ring $Hom(G, Z(G))$ is radical if and only if G is purely non-abelian.*

The above theorem can be generalized to arbitrary finite rings as follows.

Theorem 2.4. *Let R be a finite ring. Then R is radical if and only if 0 is the only idempotent in R .*

Let be $R = Hom(G, Z(G))$. We have R is non-radical if and only if there exists a non-zero idempotent homomorphism $e : G \rightarrow Z(G)$, and clearly this is equivalent to the existence of a non-trivial abelian direct factor of G .

The proof of Theorem 2.4 is based on the following result.

Lemma 2.5. *Let x be an element of a semigroup S such that $x^n = x^m$ for some positive integers $n \neq m$. Then the set $\{x^k \in S, k > 0\}$ contains an idempotent.*

Proof. For every $n > 0$, let $[n] = \{k > 0, x^k = x^n\}$.

Assume that $n < \min[2n]$, for all $n > 0$. There exist by assumption $n < m$ such that $x^n = x^m$, so the class $[n]$ is unbounded since $n + k(m - n) \in [n]$, for all $k > 0$. On the other hand if $l \in [n]$, then $2n \in [2l]$, and so $l < 2n$, a contradiction.

Hence, there exists n such that $n_0 = \min[2n] \leq n$. If $n_0 = n$, then x^n is an idempotent element of S . And if $n_0 < n$, then x^{2n-n_0} is an idempotent, since

$$(x^{2n-n_0})^2 = x^{4n-2n_0} = x^{2n} x^{2n-2n_0} = x^{n_0} x^{2n-2n_0} = x^{2n-n_0}.$$

The result follows. \square

Proof of Theorem 2.4. Suppose that R is not radical. Since R° contains every nilpotent element, then R contains a non-nilpotent element x . And since R is finite, the set of all the powers of x can not be infinite. Hence there exist $n \neq m$ such that $x^n = x^m$. The existence of a non-zero idempotent element follows now from Lemma 2.5.

Conversely, if $x \neq 0$ is an idempotent of R , then $-x \notin R^\circ$. Otherwise there exists an element $y \in R$ such that $-x + y - xy = 0$, if we multiply this equation by x on the left we obtain $-x = 0$, which is not the case. Hence $R^\circ \neq R$, and so R is not radical. The result follows. \square

As an immediate consequence of Theorem 2.3, we have

Corollary 2.6. *If G is a purely non-abelian finite group, then the ring $\text{Hom}(G, Z(G))$ is nilpotent. In particular, every homomorphism $h : G \rightarrow Z(G)$ is nilpotent.*

The following corollary is well-known in the litterature (see [9]).

Corollary 2.7. *The central automorphism group of a purely non-abelian finite group is nilpotent.*

We can also bound the rank of $\text{Aut}_c(G)$ using Lemma 1.3.

Corollary 2.8. *Let G be a purely non-Abelian finite group. Then $r(\text{Aut}_c(G)) \leq 3r(R^+)$, where R denotes the ring $\text{Hom}(G, Z(G))$. The bound 3 can be replaced by 2 if the order of $Z(G)$ is odd. In partucular if G is a p -group then, $r(\text{Aut}_c(G)) \leq 2d(G)d(Z(G))$ for $p > 2$, and $r(\text{Aut}_c(G)) \leq 3d(G)d(Z(G))$ for $p = 2$.*

Proof. The first part follows from Lemma 1.3. For the second observe that every homomorphism $h : G \rightarrow Z(G)$ can be factorized on G' , this induces an isomorphism between the two groups $\text{Hom}(G, Z(G))$ and $\text{Hom}(G/G', Z(G))$. The result follows now from Lemma 1.1. \square

3. Adjoint groups of p -nil rings

In this section we investigate more closely the structure of $\text{Aut}_c(G)$ when G is a finite p -group with $Z(G) \leq \Phi(G)$. This situation motivates the introduction of the following notions.

Definition 3.1. *Let p be a prime number and R be a ring. We say that R is left (right, resp) p -nil if every element x of order p in R^+ is a left (right, resp) annihilator of R , that is $px = 0$ implies $xy = 0$ ($yx = 0$, resp), for all $y \in R$. The ring R is said to be p -nil if it is left and right p -nil.*

For instance, the subring $S = pR$ of any ring R is p -nil. Also we check easily that the left and the right annihilators of $\Omega_1(R^+)$ are respectively right and left p -nil.

The following theorems shed some lights on the structure of the adjoint groups of these rings.

Theorem 3.2. *Let R be a ring with an additive group of finite exponent p^m . If R is left or right p -nil, then R is nilpotent of class at most m . In particular the adjoint group R° is nilpotent of class at most m .*

Proof. Assume that R is left p -nil. We proceed by induction on n to prove that $p^{m-n+1}R^n = 0$. This is obvious for $n = 1$. Now if $x \in R^n$, then by induction $p^{m-n+1}x = 0$. It follows that $p^{m-n}x$ has order 1 or p , therefore $(p^{m-n}x)y = p^{m-n}(xy) = 0$, for all $y \in R$. This shows that $p^{m-n}R^{n+1} = 0$. Now, for $n = m + 1$ we have $R^{m+1} = 0$, this prove that R is nilpotent of class at most m . The result follows for R right p -nil by a similar argument. The second assertion follows from Lemma 1.2. \square

Lemma 3.3. *If R is a left (right, resp) p -nil ring, then the factor ring $R/\Omega_n(R)$ is left (right, resp) p -nil for all $n \geq 1$, where $\Omega_n(R)$ denotes the ideal $\{x \in R, p^n x = 0\}$.*

Proof. Assume that R is left p -nil, and let be $\bar{x} \in R/\Omega_n(R)$ such that $p\bar{x} = \bar{0}$. Then $px \in \Omega_n(R)$, so $p^n x \in \Omega_1(R)$, and by assumption $(p^n x)y = p^n(xy) = 0$, for all $y \in R$. This shows that $xy \in \Omega_n(R)$, for all $y \in R$, that is \bar{x} is a left annihilator of $R/\Omega_n(R)$. The result follows for R right p -nil by a similar argument. \square

Theorem 3.4. *Let R be a p -ring, p odd. If R is left or right p -nil, then $\Omega_{\{n\}}(R^\circ) = \Omega_n(R)$, for every $n \geq 1$. In particular we have $\Omega_n(R^\circ) = \Omega_{\{n\}}(R^\circ)$.*

Proof. We denote by $x^{(k)}$ the k th power of x in the adjoint group of R .

For $n = 1$ we have, if $px = 0$ then $x^i = 0$ for $i \geq 2$. Hence

$$x^{(p)} = \sum_{i \geq 1} \binom{p}{i} x^i = px = 0,$$

and so $x \in \Omega_{\{1\}}(R^\circ)$. Conversely, if $x^{(p)} = 0$ then

$$px = - \sum_{i \geq 2} \binom{p}{i} x^i.$$

Let p^m be the additive order of x . If $m \geq 2$, then $p^{m-1}x$ has order p , hence $p^{m-1}x^2 = 0$, and similarly we obtain $p^{m-2}x^i = 0$, for $i \geq 3$. Now if we multiply the above equation by p^{m-2} we obtain

$$p^{m-1}x = - \sum_{i \geq 2} \binom{p}{i} p^{m-2}x^i = 0$$

This contradicts the definition of the order of x . Therefore $m \leq 1$, and so $x \in \Omega_1(R)$.

Now we proceed by induction on n . If $x \in \Omega_n(R)$, then $px \in \Omega_{n-1}(R)$. This implies that $x + \Omega_{n-1}(R) \in \Omega_1(R/\Omega_{n-1}(R))$. Lemma 3.3 and the first step imply that $x + \Omega_{n-1}(R) \in \Omega_{\{1\}}((R/\Omega_{n-1}(R))^\circ)$. Hence $x^{(p)} \in \Omega_{n-1}(R)$, and by induction $x^{(p)} \in \Omega_{\{n-1\}}(R^\circ)$. Thus $x \in \Omega_{\{n\}}(R^\circ)$. It follows that $\Omega_n(R) \subset \Omega_{\{n\}}(R^\circ)$. The inverse inclusion follows similarly.

Finally, the equality $\Omega_n(R^\circ) = \Omega_{\{n\}}(R^\circ)$ follows from the fact that $(\Omega_n(R))^\circ$ is a subgroup of R° and $\Omega_n(R^\circ)$ is generated by $\Omega_{\{n\}}(R^\circ)$. \square

Corollary 3.5. *Let R be a p -ring, p odd. If R is p -nil, then $\Omega_1(R^\circ) \leq Z(R^\circ)$, in other word R° is p -central.*

Proof. Every element x of $\Omega_1(R^\circ)$ lies $\Omega_1(R)$ by the above theorem. Hence x is an annihilator of R , and so it lies in the center of R° . \square

Note that this can be used to prove Lemma 1.3 among the same lines of Dickenschied proof ([6]), only we use the fact that the group $(pR)^\circ$ is p -central instead of being powerful (a finite p -group G is powerful if G/G^p (G/G^4 , for $p = 2$) is abelian), and the fact that the rank of a p -central finite p -group G is bounded by $d(Z(G))$ by a result of Thompson (see [7, III, Hilfssatz 12.2]). It seems that this alternative proof is simpler, since it is easier to prove that $(pR)^\circ$ is p -central than proving that is powerful, but unfortunately this proof does not deal with the prime $p = 2$.

In connection with central automorphisms we have

Proposition 3.6. *If G is a finite p -group such that $Z(G) \leq \Phi(G)$, then the ring $\text{Hom}(G, Z(G))$ is right p -nil.*

Proof. Let be $k, h \in \text{Hom}(G, Z(G))$ such that $ph = 0$. Then $h : G \rightarrow \Omega_1(Z(G))$. Since the image of h is an elementary abelian p -group, its kernel contains the frattini subgroup, and since $Z(G) \leq \Phi(G)$ we have $kh(x) = h(k(x)) = 1$, for all $x \in G$. It follows that h is a right annihilator of the ring $\text{Hom}(G, Z(G))$. \square

The above proposition leads to a new proof of Theorem 4.8 in [9].

Corollary 3.7. *If G is a finite p -group such that $Z(G) \leq \Phi(G)$, then $\text{Aut}_c(G)$ is nilpotent of class at most $\min\{r, s\}$, where $\exp(G/G') = p^r$ and $\exp(Z(G)) = p^s$.*

Proof. By Theorem 3.2 the nilpotency class of $\text{Aut}_c(G)$ does not exceed m , where p^m is the exponent of $\text{Hom}(G, Z(G)) \cong \text{Hom}(G/G', Z(G))$ which is equal to $p^{\min\{r, s\}}$ by Lemma 1.1. \square

Theorem 3.8. *If G is a finite p -group with p odd, such that $Z(G) \leq \Phi(G)$, then*

$$\Omega_n(\text{Aut}_c(G)) = \Omega_{\{n\}}(\text{Aut}_c(G)) = \text{Aut}_{Z_n}(G)$$

where Z_n denotes the subgroup $\Omega_n(Z(G))$.

Proof. This is an immediate consequence of Theorem 3.4 and Proposition 3.6. \square

4. Non-inner central automorphisms of order p .

A longstanding conjecture asserts that every non-abelian finite p -group has a non-inner automorphism of order p . More informations about this conjecture can be found for instance in [1].

First, note that we can reduce it to indecomposable p -groups.

Proposition 4.1. *Let G be a non-abelian finite p -group. If G is decomposable then G has a non-inner central automorphism of order p .*

Proof. Assume that G is a direct product of G_1 and G_2 , where G_1, G_2 are non-trivial normal subgroups of G . Let M be a maximal subgroup of G_1 and $g \in G_1 - M$, clearly every element of G can be written in the form xg^i , where $x \in MG_2$. If z is a central element of order p in G_2 , then the mapping $xg^i \mapsto xg^iz^i$ is a central automorphism of G of order p which is not inner since it maps $g \in G_1$ to $gz \notin G_1$. \square

For p odd, the results of the previous section allows us to characterize the p -groups in which every central automorphism of order p is inner. Let denote $d = d(G)$, $d_1 = d(Z(G))$ and $d_2 = d(Z(\text{Inn}(G)))$.

Theorem 4.2. *Let G be a finite non-abelian p -group, p odd. In order for G to have a non-inner central automorphism of order p it is necessary and sufficient that $d_2 \neq d \cdot d_1$.*

For instance, the p -groups of maximal class satisfy this condition, as well as the class of non-abelian finite p -central p -groups, this follows easily from [7, III, Hilfssatz 12.2]. We need the following two lemmas to prove Theorem 4.2.

Lemma 4.3. *If G is a purely non-abelian finite p -group, then $\Omega_1(Z(G)) \leq \Phi(G)$. In particular if $\exp(Z(G)) = p$ then $Z(G) \leq \Phi(G)$.*

Proof. Let $z \in \Omega_1(Z(G))$. If there exists a maximal subgroup M such that $z \notin M$, then $G \cong \langle z \rangle \times M$. Thus G is not purely non-abelian.

This is another proof based on the nilpotency of the ring $\text{Hom}(G, Z(G))$. Let be $z \in \Omega_1(Z(G))$. To each homomorphism $r : G \rightarrow \mathbb{Z}_p$ we can associate an endomorphism $h \in \text{Hom}(G, Z(G))$ by setting $h(x) = r(x)z$, for all $x \in G$. This implies that $h^n(z) = r(z)^nz$. By corollary 2.6, h is nilpotent, so there exists an integer n such that $r(z)^n = 0$. Therefore $r(z) = 0$, since \mathbb{Z}_p is a field. This shows that z lies in the intersection of the set of all kernels of homomorphisms from G to \mathbb{Z}_p . Since every maximal subgroup of G occurs as a kernel of some homomorphism $r : G \rightarrow \mathbb{Z}_p$. It follows that $z \in \Phi(G)$. \square

Lemma 4.4. *Let G be a finite p -group. Then every inner automorphism which is central of order p is induced by some non-trivial homomorphism $h : G \rightarrow \Omega_1(Z(G))$. Moreover, if G is purely non-abelian then for every non-trivial homomorphism $h : G \rightarrow \Omega_1(Z(G))$, the order of the central automorphism $\sigma = 1_G + h$ induced by h is equal to p .*

Proof. Let be τ an inner central automorphism of order p . We can write $\tau = 1_G + h$, for some $h \in \text{Hom}(G, Z(G))$, and 1_G denotes the identity map of G . We have $h = h\tau = h + h^2$, and so $h^2 = 0$. This implies that $1_G = \tau^p = 1_G + ph$, and so $ph = 0$. Therefore $h : G \rightarrow \Omega_1(Z(G))$.

Assume that G is purely non-abelian. Since the kernel of every homomorphism $h : G \rightarrow \Omega_1(Z(G))$ contains $\Phi(G)$, Lemma 4.3 implies that $h^2 = 0$. Therefore, if $\sigma = 1_G + h$ then $\sigma^p = 1_G + \sum_{i=1}^p \binom{p}{i} h^i = 1_G + ph = 1_G$. The result follows. \square

Proof of Theorem 4.2. Suppose that G has a non-inner central automorphism $\sigma = 1_G + h_\sigma$ of order p . Let I be the image of $\Omega_1(Z(\text{Inn}(G)))$ by the Adney-Yen map. By Lemma 4.4 I is a subspace of the \mathbb{Z}_p -vector space $\text{Hom}(G, \Omega_1(Z(G)))$ of dimension d_2 . If $d_2 = d \cdot d_1$, then $I = \text{Hom}(G, \Omega_1(Z(G)))$. If $Z(G) \not\leq \Phi(G)$ then we can find an element $g \in Z(G) - M$ for some maximal subgroup M of G . Consider a non-trivial element $z \in \Omega_1(Z(G)) \cap M$ and let $h(x) = z^{r(x)}$, where $r : G \rightarrow \mathbb{Z}_p$ is the

homomorphism defined by $r(mg^i) = i \pmod p$, $m \in M$. Clearly, $h \in I$ and $1_G + h$ is not inner, since it maps g to gz , a contradiction. It follows that $Z(G) \leq \Phi(G)$. Theorem 3.8 implies that $ph_\sigma = 0$, that is $h_\sigma \in I$. It follows that $\sigma = 1_G + h_\sigma$ is inner, a contradiction. Therefore $d_2 \neq d \cdot d_1$.

Conversely, by Proposition 4.1 we may suppose that G is purely non-abelian. If $d_2 \neq d \cdot d_1$, then I is a proper subspace of $\text{Hom}(G, \Omega_1(Z(G)))$. Hence there exists $h : G \rightarrow \Omega_1(Z(G))$ such that the automorphism $\sigma = 1_G + h$ is not inner. It follows from Lemma 4.4 that σ has order p . \square

Let G be a finite non-abelian p -group of order p^n and class c . Under the above notation, does the equality $d_2 = d \cdot d_1$ imply that G has a cyclic center?

Assume that G is a counter example to this question, by a formula of Abdollahi [1, Theorem 2.5] we have $d_1 \cdot (d + 1) \leq r + 1$, where $r = n - c$ is the coclass of G . The class of G must be ≥ 3 , otherwise we would have $d_2 = d(G/Z(G)) \leq d$, so $d_1 = 1$ which is not the case. On the other hand $d \geq 2$, hence $3d_1 \leq r + 1$, so we must have $r + 1 \geq 6$, thus $n \geq 5 + c \geq 8$.

This shows that if a counter example to the above question exists then it has at least coclass 5 and order p^8 . It is well-known that in a powerful p -group G , every subgroup can be generated by $d(G)$ elements, so a counter example to our question can not be a powerful p -group.

Acknowledgment. The first author is grateful to Miloud Reguiat for his encouragement, and his comments about early drafts of this paper.

REFERENCES

- [1] A. Abdollahi, Powerful p -groups have non-inner automorphisms of order p and some cohomology, *J. Algebra*. **323** (2010), 779-789.
- [2]
- [3] A. Abdollahi, Powerful p -groups have non-inner automorphisms of order p and some cohomology, *J. Algebra*. **323** (2010), 779-789.
- [4] J.E. Adney and T. Yen, Automorphisms of a p -group, *Illinois J. Math.* **9** (1965), 137-143.
- [5] M.S. Attar, Finite p -groups in which each central automorphism fixes centre elementwise, *Comm. Algebra*. **40** (2012), 1096-1102.
- [6] M.J. Curran, Finite groups with central automorphism group of minimal order, *Math. Proc. Royal Irish Acad.* **104 A(2)** (2004), 223-229.
- [7] M.J. Curran and D.J. McCaughan, Central automorphisms that are almost inner, *Comm. Algebra*. **29 (5)** (2001), 2081-2087.
- [8] O. Dickenschied, On the adjoint group of some radical rings, *Glasgow Math. J.* **39** (1997), 35-41.
- [9] B. Huppert. Endliche Gruppen. I. *Die Grundlehren der Mathematischen Wissenschaften*, Band 134. Springer-Verlag, Berlin, 1967.
- [10] R.L. Kruse and D.T. Price, *Nilpotent rings*, Gordon and Breach, New York (2010).
- [11] M.H. Jafari and A.R. Jamali, On the nilpotency and solubility of the central automorphism group of finite group, *Algebra Coll.* **15:3** (2006), 485-492.
- [12] M.K. Yadav, On central automorphisms fixing the center element-wise, *Comm. Algebra*. **37** (2009), 4325-4331.

Yassine Guerboussa

Department of Mathematics, University Kasdi Merbah Ouargla, Ouargla, Algeria

Email: yassine_guer@hotmail.fr

Bounabi Daoud

Department of Mathematics, University of Setif , Setif, Algeria

Email: boun_daoud@yahoo.com